Enhancement of Coherent Response by Quenched Disorder

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We investigate the effects of quenched disorder on the coherent response in a driven system of coupled oscillators. In particular, the interplay between quenched noise and periodic driving is explored, with particular attention to the possibility of resonance. The phase velocity is examined as the response of the system; revealed is the enhancement of the fraction of oscillators locked to the periodic driving, displaying resonance behavior. It is thus concluded that resonance behavior may also be induced by quenched disorder which does not have time-dependent correlations.

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In recent years, the *stochastic resonance* (SR) has drawn much attention, which stands for the phenomena that the response of a system to a periodic driving force is enhanced by an appropriate amount of noise rather than suppressed [1, 2, 3]. Those SR phenomena observed in various real systems [4, 5, 6] are known to occur through the cooperative interplay between the noise and the external driving force, and require three ingredients: (i) an energetic activation barrier; (ii) a weak input such as periodic driving; (iii) a random noise [3]. In particular, the SR may be understood in terms of matching of two time scales, the period of the driving force and the inverse of the Kramers hopping rate associated with the random noise. Note that the noise here usually has time-dependent correlations. Namely, the correlation between two noise forces η_i and η_j is given by $\langle \eta_i(t)\eta_j(t')\rangle = 2T\delta_{ij}\delta(t-t')$, where T represents the noise strength. Here naturally arises a question how does the SR behavior depends on such characteristics of the noise. In particular, one may ask whether such resonance behavior appears even for the quenched noise without time-dependent correlations.

To resolve this, we consider the system of coupled oscillators described by

$$\dot{\varphi}_i = \omega_i - \frac{K}{N}\sum_{j=1}^{N} \sin(\varphi_i - \varphi_j) + I_i \cos\Omega t, \qquad (1)$$

where φ_i represents the phase of the ith limit-cycle oscillator. The first term ω_i on the right-hand side denotes the intrinsic frequency of the ith oscillator, assumed to be distributed according to the Gaussian distribution $g(\omega)$ with zero mean ($\langle\omega\rangle = 0$) and correlations $\langle \omega_i\omega_j\rangle = D\delta_{ij}$. Note that correlations between ω_i's are not time-dependent but quenched in time, in sharp contrast with the usual thermal noise. The second term on the right-hand side describes the coupling between oscillators, whereas the third term represents the external periodic driving with the driving strength I_i chosen from a certain distribution function $f(I)$. In the absence of the external driving ($I_i = 0$), Eq. (1) describes the well-known Kuramoto model, where the synchronization phenomena have been investigated extensively [7]. The scattered intrinsic frequency or quenched noise ω_i competes with the coupling strength K in the system: When the coupling is strong enough to overcome the scatteredness of intrinsic frequencies, synchronization emerges. Such synchronization behavior may be explored by measuring the order parameter

$$\Delta \equiv \left\langle \left| \frac{1}{N}\sum_j e^{i\varphi_j} \right| \right\rangle, \qquad (2)$$

where nonzero Δ implies the emergence of synchronization. It is known that the synchronization behavior of the system is characterized by $\Delta \sim (K - K_c)^\beta$ with $\beta = 1/2$ near the critical coupling strength $K_c = 2/\pi g(0)$ [7]. The Kuramoto model has been also extended by means of introducing a constant in the argument of sine coupling [8]. Such extended model is also studied in Ref. [9], where frequency locking in the system of Josephson arrays is investigated.

Meanwhile, when the external periodic driving comes into system, the synchronization behavior has been observed to appear periodically [10]. However, the possibility of the resonance phenomena, which may occur via the cooperative interplay between the external driving and the quenched noise, has not been investigated. In this work, we examine the linear response of the system, with particular attention to the possibility of the resonance behavior due to the quenched disorder.

We first investigate analytically the dynamics of the system governed by Eq. (1). The order parameter Δ defined in Eq. (2) allows us to reduce Eq. (1) into a single equation

$$\dot{\varphi} = \omega - K\Delta \sin\varphi + I\cos\Omega t, \qquad (3)$$

where Δ is to be determined by imposing self-consistency, and indices are suppressed for simplicity. Equation (3) reminds one of the single resistively shunted Josephson junction under combined direct and alternating currents. It is well known that such a system can be locked to the external driving, which is characterized by the Shapiro

step [11]

$$\frac{\bar{\dot{\varphi}}}{\Omega} = n \tag{4}$$

with n integer, where $\bar{\dot{\varphi}}$ ($\equiv v$) is the time-averaged phase velocity. Such mode locking features suggest the ansatz

$$\varphi = \varphi_0 + n\Omega t + \sum_{p=1}^{\infty} A_p \sin(p\Omega t + \alpha_p) \tag{5}$$

for the locked phase of the oscillator on the nth step. The external periodic driving in the system also leads us to expect periodic synchronization [10], where the order parameter Δ is decomposed as

$$\Delta = \Delta_0 + \sum_{q=1}^{\infty} \Delta_q \cos(q\Omega t + \beta_q). \tag{6}$$

Inserting Eqs. (5) and (6) into Eq. (3), we obtain

$$\begin{aligned} n\Omega + \sum_{p=1}^{\infty} p\Omega A_q \cos(p\Omega t + \alpha_p) - I \cos\Omega t \\ = \omega - K\left[\Delta_0 + \sum_{q=1}^{\infty} \Delta_q \cos(q\Omega t + \beta_q)\right] \\ \times \left(\prod_{r=1}^{\infty} \sum_{\ell_r=-\infty}^{\infty} J_{\ell_r}(A_r)\right) \sin\Phi, \end{aligned} \tag{7}$$

where $J_{\ell_r}(x)$ is the ℓ_rth Bessel function and $\Phi \equiv \varphi_0 + \sum_{s=1}^{\infty} \ell_s \alpha_s + (n + \sum_{s=1}^{\infty} s\ell_s)\Omega t$. The integers ℓ_s satisfying $\sum_s s\ell_s = -n$ contribute to the dc component in Φ:

$$n\Omega = \omega - K\Delta_0 \left(\prod_{r=1}^{\infty} \sum_{\ell_r}{}' J_{\ell_r}(A_r)\right) \sin\tilde{\Phi} + \mathcal{O}(K\Delta_1), \tag{8}$$

where $\tilde{\Phi} \equiv \varphi_0 + \sum_{s=1}^{\infty} \ell_s \alpha_s$ and the prime in the summation stands for the constraint $\sum_s s\ell_s = -n$. This gives an estimation of the dc driving strength ω corresponding to the integer locking. The amplitude A_p and phase α_p of the ac component with frequency $p\Omega$ can be determined from the equation

$$\begin{aligned} &- p\Omega A_p \cos(p\Omega t + \alpha_p) + \delta_{p,1} I \cos p\Omega t \\ &= K\Delta_0 \left(\prod_{r=1}^{\infty} \sum_{\ell_r^+} J_{\ell_r^+}(A_r)\right) \sin\left(\varphi_0 + p\Omega t + \sum_{s=1}^{\infty} \ell_s^+ \alpha_s\right) \\ &+ K\Delta_0 \left(\prod_{r=1}^{\infty} \sum_{\ell_r^-} J_{\ell_r^-}(A_r)\right) \sin\left(\varphi_0 - p\Omega t + \sum_{s=1}^{\infty} \ell_s^- \alpha_s\right) \\ &+ K\Delta_p \cos(p\Omega t + \beta_p) \left(\prod_{r=1}^{\infty} \sum_{\ell_r}{}' J_{\ell_r}(A_r)\right) \sin\tilde{\Phi} \end{aligned} \tag{9}$$

with integers ℓ_r^+ and ℓ_r^- satisfying $\sum_{s=1}^{\infty} s\ell_s^+ = p - n$ and $\sum_{s=1}^{\infty} s\ell_s^- = -p - n$, respectively. When $K\Delta_0$ is sufficiently small compared with the driving amplitude, Eq. (9) with $p = 1$ yields A_1 and α_1 to the zeroth order in $K\Delta_0$: $A_1 = I/\Omega$ and $\alpha_1 = 0$. It is observed numerically that the dc component Δ_0 of the order parameter is much larger than higher-order (ac) components [$\Delta_0 \gg \Delta_\ell (\ell \geq 1)$], which allows the expansion of the locked phase to the zeroth order in $K\Delta_0$. This gives the (locked) phase of the oscillator on the nth step as

$$\varphi = \varphi_0 + n\Omega t + \frac{I}{\Omega} \sin\Omega t + \mathcal{O}(K\Delta_0 I/2\Omega^2), \tag{10}$$

which yields the range of locked oscillators:

$$\omega - n\Omega = K\Delta_0 (-1)^n J_n(I/\Omega) \sin\phi_0 + \mathcal{O}(K\Delta_1). \tag{11}$$

It implies that the oscillators in the range

$$n\Omega - K\Delta_0 |J_n(I/\Omega)| \leq \omega \leq n\Omega + K\Delta_0 |J_n(I/\Omega)| \tag{12}$$

display the locking behavior ($v/\Omega = n$), with the higher-order terms of $\mathcal{O}(K\Delta_1)$ neglected.

The quenched disorder ω and the driving amplitude I are chosen from the distribution function $g(\omega)$ and $f(I)$, respectively, which yields the fraction r_n of the oscillators locked to the nth step:

$$r_n = \int_{-\infty}^{\infty} f(I) dI \int_{n\Omega - K\Delta_0 |J_n(I/\Omega)|}^{n\Omega + K\Delta_0 |J_n(I/\Omega)|} g(\omega) d\omega. \tag{13}$$

For the Gaussian distribution $g(\omega) = (1/\sqrt{2\pi D}) e^{-\omega^2/2D}$ and the delta function one $f(I) = (1/2)[\delta(I - I_0) + \delta(I + I_0)]$, the fraction r_n in Eq. (13) is given by

$$\begin{aligned} r_n \;=\; & \frac{1}{2}\left[\mathrm{erf}\left(\frac{n\Omega + K\Delta_0 |J_n(I_0/\Omega)|}{\sqrt{2D}}\right)\right. \\ & \left. -\; \mathrm{erf}\left(\frac{n\Omega - K\Delta_0 |J_n(I_0/\Omega)|}{\sqrt{2D}}\right)\right], \end{aligned} \tag{14}$$

where $\mathrm{erf}(x)$ denotes the error function. To estimate the behavior of the fraction r_n as the variance D varies, we should know the behavior of the dc component Δ_0 as a function of the variance D. Note that the self-consistency equation for the order parameter gives the behavior of Δ_0 only near the critical point. To obtain Δ_0 in the whole range of the disorder strength, we resort to numerical simulations and integrate Eq. (1) via Heun's method [12] with the discrete time step $\delta t = 0.01$. While the equations of motion are integrated for $N_t = 8 \times 10^4$ time steps, the data from the first $N_t/2$ steps are discarded in measuring quantities of interest. The system size N has been considered up to $N = 20000$, so that no appreciable size-dependence is observed. The driving amplitude I_0 in the distribution $f(I)$ and the driving frequency Ω have been chosen to be $I_0 = 0.8$ and $\Omega = 1.0$, 1.2, and 1.4, respectively; the coupling strength K has been set equal

to unity for convenience. We measure the order parameter Δ, and obtain the component Δ_0 by taking the time average.

The dc component Δ_0 is shown in the inset of Fig. 1 as a function of the disorder strength D, displaying the monotonically decreasing behavior. Using this, we compute the fraction $r_{\pm 1}$ of the oscillators which are locked to the first step since $r_{\pm 1}$ is most dominant over other components $[r_\ell(\ell \geq 2)]$. Figure 1 displays the total fraction $r_1 + r_{-1}$ for $K = 1.0$, $I_0 = 0.8$, and $\Omega = 1.0, 1.2, 1.4$ versus the disorder strength D. It is found that the fraction first increases with D, which implies that a larger number of oscillators tends to follow the driving force as the quenched disorder becomes stronger. Remarkably as the disorder is increased further, the fraction reaches the maximum and begins to decrease. For example, the optimal disorder strength is observed to be $D_m \approx 0.17$ for $\Omega = 1.0$. Such behavior of the fraction $r_{\pm 1}$ is reminiscent of the SR, which is known to occur through the cooperative interplay between the external periodic driving force and the random noise. Note that in sharp contrast to the random noise in the conventional SR, the noise in the system governed by Eq. (1) is quenched with no time-dependent correlations. Namely, here the quenched disorder enhances the coherent response of the system. It is also shown that as the driving frequency Ω is raised, the fraction $r_{\pm 1}$ of locked oscillators diminishes while the optimal strength D_m shifts to larger values. These tendencies reflect that the oscillators are reluctant to follow the driving which changes too fast, and that the disorder strength D should be enlarged to fit the high driving frequency. Another point is that the fraction r_0 of the oscillators locked to the $n = 0$ step does not exhibit such resonance behavior; it rather displays the monotonic decreasing behavior, which is quite similar to that of the dc component Δ_0 of the phase order parameter. Higher fractions for $n \geq 2$ also show the resonance behavior, although the magnitude is too small to be clearly discriminated. We have also investigated the role of coupling on the enhancement of coherent response. As the coupling strength K decreases, such resonance behavior is found to be suppressed.

To confirm such resonance behavior, we now perform numerical simulations. We obtain the probability distribution $P(v)$ for the time-averaged phase velocity v under the same conditions as before, varying the disorder strength D. The data are then averaged over 100 independent sets of $\{\omega_i\}$. The inset of Fig. 2 displays the probability distribution for $\Omega = 1.0$ and $D = 0.20$, where three sharp peaks appear at $v = 0$ (not fully shown) and $\pm\Omega$. The two peaks at the driving frequency $(\pm\Omega)$ manifest that some fraction of the oscillators follow the external driving force, locked to the latter. Note that such peaks may also appear at higher frequencies $n\Omega$ $(|n| \geq 2)$, although they are too small to be manifestly shown.

In the absence of the quenched disorder $(D = 0)$, those

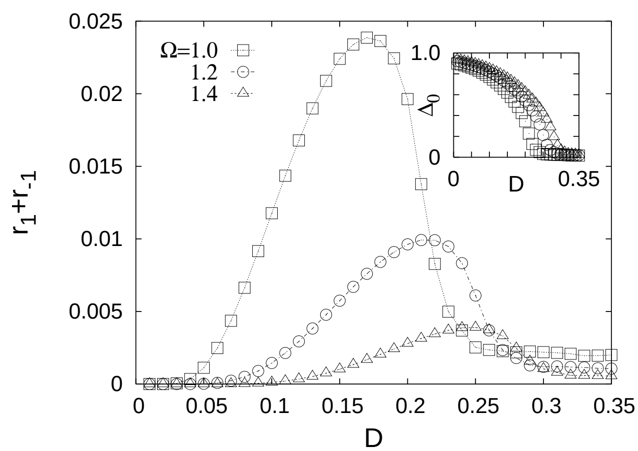

FIG. 1: Total fraction $r_1 + r_{-1}$ of the oscillators locked to the first step $(n = \pm 1)$ versus the disorder strength D for $K = 1.0$, $I_0 = 0.8$, and $\Omega = 1.0$, 1.2, and 1.4. Inset: dc component Δ_0 in the system of size $N = 20000$ at various driving frequencies.

peaks at $v = \pm\Omega$ do not emerge, and only the delta peak appears at $v = 0$. As the disorder strength is increased further, on the other hand, peaks show up at the driving frequency, with the height growing. The peak height $h_{\pm 1}$ at the driving frequency may be regarded as a good indicator which describes the response of the system to the external driving.

We numerically obtain the peak height, subtracting the background noise given by the average of ten data points around the peak. Figure 2 displays the total height $h_1 + h_{-1}$ versus the disorder strength D in the system of size $N = 20000$, with the driving frequency varied from $\Omega = 1.0$ to $\Omega = 1.4$. We have considered the system size up to $N = 40000$, where no appreciable size-dependence is observed. The total height is found to first increase with the disorder strength D and reaches its maximum at a finite value of the disorder, displaying quite a similar feature to the fraction $r_{\pm 1}$ [see Fig. 1]. The enhancement of the height indicates that the quenched disorder actually increases the number of the oscillators locked to the external driving. To our knowledge, such enhancement induced by the quenched disorder without time-dependent correlations has not been addressed before. It is observed in Fig. 2 that the increase of the driving frequency tends to suppress the total height, and to shift the optimal disorder strength to larger values, which may be related with the intrinsic time scale of the system. Note that the conventional SR phenomena have been known to occur when the Kramers hopping rate and the external driving frequency match each other. To see such time scale matching in the quenched-noise-induced resonance, we investigate the relaxation dynamics of the system, and probe the time evolution of the renormalized

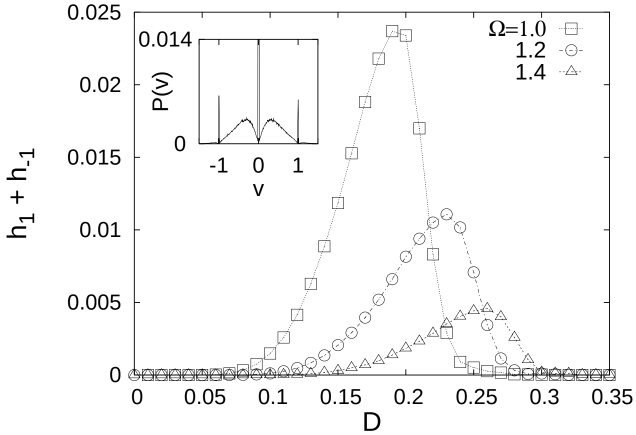

FIG. 2: Total height $h_1 + h_{-1}$ of the peak at $v = \pm\Omega$ in the probability distribution versus the disorder strength D. Inset: probability distribution $P(v)$ of the time-averaged phase velocity v for $K = 1.0,\ I_0 = 0.8,\ \Omega = 1.0$, and $D = 0.20$.

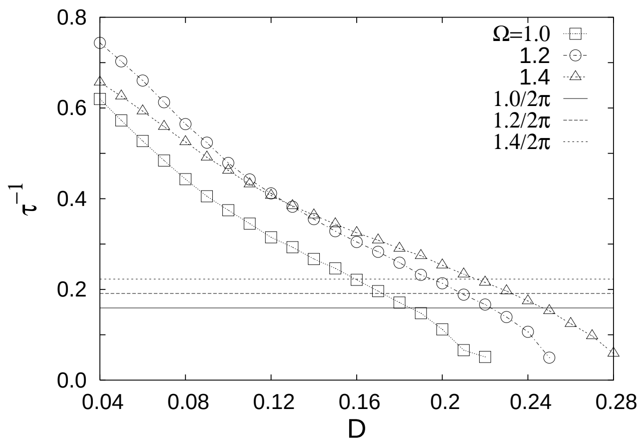

FIG. 3: Inverse relaxation time τ^{-1} vs the disorder strength D. The crossing points with the lines $1.0/2\pi, 1.2/2\pi$, and $1.4/2\pi$ yield the values of D_m at the corresponding driving frequency.

synchronization order parameter

$$\tilde{\Delta}(t) \equiv \frac{\Delta(t) - \Delta_{\rm eq}}{\Delta(0) - \Delta_{\rm eq}}, \tag{15}$$

where $\Delta_{\rm eq}$ and $\Delta(0)$ represent the equilibrium value and the initial one, respectively. The renormalized order parameter $\tilde{\Delta}(t)$ is thus expected to decay from unity ($t = 0$) to zero ($t \to \infty$): $\tilde{\Delta}(t) \sim \exp(-t/\tau)$.

Figure 3 shows the inverse relaxation time τ^{-1} versus the disorder strength D for $I_0 = 0.8$ and $\Omega = 1.0,\ 1.2$, and 1.4. The intrinsic time scale τ of the system is observed to increase indefinitely as the disorder strength D approaches the critical value D_c beyond which synchronization disappears. The horizontal lines in Fig. 3 describe the time-scale matching conditions at various driving frequencies: the value of D at the crossing point corresponds to D_m at which the enhancement of the response is maximized. For example, for $\Omega = 1.0,\ 1.2$, and 1.4, the crossing points lead to $D_m \approx 0.19,\ 0.21$, and 0.22, respectively, which are consistent with the values obtained both analytically and numerically [see Figs. 1 and 2]. We thus conclude that quenched disorder may also enhance the coherent response of the system via the mechanism of the time-scale matching.

In summary, we have explored the effects of quenched disorder on the coherent response in the driven system of coupled oscillators. We have investigated the interplay between quenched disorder and external periodic driving, with particular attention to the possibility of resonance behavior. The phase velocity is probed as the response of the system; revealed is the enhancement of the fraction of the oscillators locked to the external driving, exhibiting resonance behavior. This provides the first observation of the resonance behavior induced by quenched disorder. In a biological system such as synchronous fireflies, different firing frequencies of fireflies may be regarded as the quenched disorder. Our results are applicable to expect those different firing frequencies (instead of same ones) may enhance the coherent response of the system.

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